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The dimension of two levels of the Boolean lattice

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Abstract

Let $B(j,k;n)$ be the ordered set obtained by ordering the j element and k element subsets of an n element set by inclusion. We review results and proof techniques concerning the dimension $\dim(j,k;n)$ of $B(j,k;n)$ for various ranges of the arguments j, k, and n.

1. Introduction

Let $P=(X,<_P)$ be an ordered set. A linear extension of P is a linear order $L=(X,<_L)$ such that if $x<_P y$ then $x<_L y$. The dimension $\dim(P)$ of P is the cardinality t of the smallest collection $\Sigma=\{L_i\colon i\in[t]\}$ of linear extensions of P such that $x<_P y$ iff $x<y$ in L_i for all $i\in[t]$, i.e., if x is incomparable to y then there exist $i,j\in[t]$ such that $x<y$ in L_i and $y<x$ in L_j. Such a collection is called a *realizer* of P. In this article we review results and proof techniques concerning the dimension of ordered sets formed from two levels of the Boolean lattice. This work began in 1950 with Dushnik [2], continued in with Spencer [14] in 1971 and Füredi and Kahn [5] in 1986, and gained considerable momentum in the 1990s with the work of Brightwell, Füredi, Hurlbert, Hoşten, Kostochka, Milner, Morris, Talysheva, Trotter, and the author [1,4,6–9]. However it is far from complete.

First we need some notation. For any positive integer n, let $[n]=\{1,2,\dots,n\}$. For any set X let 2^X denote the power set of X and $\binom{X}{k}$ denote the family of k element subsets of X. For positive integers $j\leqslant k<n$, let $B(n)=(2^{[n]},\subset)$ denote the ordered set consisting of the subsets of $[n]$ ordered by inclusion and $B(j,k;n)=(\binom{[n]}{j}\cup\binom{[n]}{k},\subset)$ denote the ordered set consisting of the j and k element subsets of $[n]$ ordered by inclusion. Finally, let $\dim(j,k;n)$ denote the dimension of $B(j,k;n)$.

We shall mostly be concerned with the case when $j=1$. There are several reasons for this. First this is the best understood case. This is also the case that has had application to the general theory of ordered sets. For example, $\dim(1,k;n)$ plays a crucial role in

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bounding the dimension of an ordered set in terms of the maximum degree of its comparability graph [5]. Scarf complexes, defined and used in commutative algebra, are directly related to $\dim(1,2;n)$ [7]. For his work on infinite ordered sets Pouzet [13] needed to know $\dim(1,<\omega;\kappa)$, the dimension of the finite subsets of an infinite cardinal κ. Finally, the following natural characterization of $\dim(1,k;n)$ makes the notion appealing on purely combinatorial grounds. Let $L=(X,<_L)$ be a linear order on a set X, $S\subset X$, and $x\in X\backslash S$. We write $S<_L x$ to mean that $y<_L x$ for every $y\in S$ and $S\nless_L x$ to mean that this is not the case.

Proposition 1.1. *The dimension* $\dim(1,k;n)$ *of* $B(1,k;n)$ *is the least t such that there exists a collection* $\{L_i\colon i\in[t]\}$ *of t linear orders on* $[n]$ *with order relation* $<_i$ *such that* $(*)$ *for all* $S\subset[n]$ *and* $x\in[n]\backslash S$ *there exists* $i\in[t]$ *with* $S<_i x$.

For the rest of the article we will abuse notation by calling a collection of linear orders on $[n]$ a realizer of $B(1,k;n)$ if it satisfies $(*)$.

1.1. Organization and notation

Our treatment is organized as follows. In the next four sections we consider the case $j=1$. We start with Dushnik's exact calculations for large k. In the next section we present very tight estimates of Spencer, Trotter, Hoşten, and Morris for the case of very small k. Next we study the intermediate range, where there is still lots of work to be done. Finally we consider the case of infinite cardinals. In the last section we look at the case $j>1$, where the results are very rough.

We shall write $S=\{x_1<x_2<\cdots<x_n\}$ to indicate both the elements of S and their order. For a linear order L, we write $\max_L S$ $(\min_L S)$ to denote the maximum (minimum) element of S with respect to L. The inverse L^* of L is the order such that $x<y$ in L^* iff $y<x$ in L. For a partial order P we write $\mathrm{MAX}_P(S)$ to denote the set of maximal elements in S with respect to P. For sets A and B, let $[B]^{[A]}$ denote the set of functions from A to B. We denote the set of permutations of A by Perm (A).

1.2. Combinatorial preliminaries

A family of sets $\mathscr{F}=\{F_i\colon i\in[m]\}\subset\binom{[t]}{s}$ is a (θ,s,t)*-packing* if $|F\cap F'|<\theta$ for every pair of distinct sets F, $F'\in\mathscr{F}$. Let $\mathrm{pack}(\theta,s,t)$ be the cardinality of the largest (θ,s,t)-packing. We shall use the following simple facts.

Lemma 1.2. *Let* θ, p, s, t, *and* y *be positive integers and* Δ *be a nonnegative integer, with p prime.*

1. $\mathrm{pack}(1,s,t)=\lfloor t/s\rfloor$.
2. $\mathrm{pack}(2,y-1,\binom{y}{2})=y$.
3. $\mathrm{pack}(2,y-1+\Delta,\binom{y}{2}+y\Delta)\geqslant y$.

4. $\text{pack}(\theta,s,t) \leqslant \binom{t}{\theta}/\binom{s}{\theta}$.
5. If $k \leqslant p$, then $\text{pack}(\theta,k,kp) \geqslant p^{\theta}$.

Proof. (1) Follows from the pigeon hole principle. For (2) consider the family $\{F_i\colon i \in [y]\}$, where $F_i = \{\{i,j\}\colon j \in [y]\backslash\{i\}\}$. (3) is a combination of (1) and (2). (4) follows from the fact that every θ-subset S is contained in at most one set F of a (θ,s,t)-packing $\mathscr{F}$ and each set $F \in \mathscr{F}$ contains $\binom{s}{\theta}$ such subsets. For (5), let A be a k-subset of the finite field F of order p. Consider the family $\{F_f\colon f \in F[x] \text{ and } \deg(f) < \theta\}$, where $F_f = \{(a, f(a))\colon a \in A\}$. $\square$

We will also use the following Lemma of Kleitman and Markowsky.

Lemma 1.3 (Kleitman and Markowsky). *The Boolean lattice $B(t)$ has at least $2^{\binom{t}{\lfloor t/2 \rfloor}}$ and at most $2^{(1+O((\log t)/t))\binom{t}{\lfloor t/2 \rfloor}}$ antichains.*

2. Large k

In this section we will calculate exact values for $\dim(1,k;n)$ in the case that $k \geqslant 2\sqrt{n}-2$. These values are due to Dushnik [2]. The techniques we develop for proving lower and upper bounds on $\dim(1,k;n)$ will be used with less precision in later sections.

2.1. Lower bounds

To prove lower bounds of the form $\dim(1,k;n) > t$, it suffices to show that for any collection of t linear of orders Σ on $[n]$, there exists a pair (x, W) with $W \in \binom{[n]}{k}$ and $x \in [n]\backslash W$ such that $W \not< x$ in L for all $L \in \Sigma$.

Theorem 2.1. *Suppose that there exist positive integers s and t such that $s+t \leqslant n$ and $\lfloor t/s \rfloor + s - 1 \leqslant k$. Then $t < \dim(1,k;n)$*

Proof. Let Σ be a t-set of linear extensions of $[n]$ and let $T = \{\max_L[n]\colon L \in \Sigma\}$. Let S be an s-subset of $[n]\backslash T$. Since $s+t \leqslant n$, $[n]\backslash T$ contains an s-subset S. By the pigeon hole principle, there exists $x \in S$ such that $|\{L \in \Sigma\colon x = \max_L S\}| \leqslant \lfloor t/s \rfloor$. Let $\Sigma_x = \{L \in \Sigma\colon x = \max_L S\}$ and $T_x = \{\max_L[n]\colon L \in \Sigma_x\}$. Finally set $W = T_x \cup (S\backslash\{x\})$. Then $|W| \leqslant \lfloor t/s \rfloor + s - 1 \leqslant k$. If $L \in \Sigma_x$, then $T_x \not\leqslant x$ in L. If $L \in \Sigma\backslash\Sigma_x$, then $S\backslash\{x\} \not\leqslant x$ in L. Thus Σ is not a realizer, since $W \not\leqslant x$ in L for all $L \in \Sigma$. $\square$

Corollary 2.2. *If there exists a natural number m such that $\lfloor n/(m+1) \rfloor + m - 1 \leqslant k$, then $n - m \leqslant \dim(1,k;n)$.*

Proof. Apply Theorem 2.1 with $s = m+1$ and $t = n-m-1$. $\square$

Corollary 2.3. *If* $k \leqslant 2\sqrt{n} - 1$, *then* $k^2/4 < \dim(1,k;n)$.

Proof. Apply the theorem with $s = \lceil k/2 \rceil$. $\square$

2.2. Upper bounds

Suppose we want to prove bounds of the form $\dim(1,k;n) \leqslant t$ by constructing a realizer Σ. The maximum element of any linear order $L \in \Sigma$ is special, since it is already over all k-subsets in L and so it is safe to put it at the bottom of all the other linear extensions in Σ. Let $m = n - t$. We will construct Σ so that $T = [n] - [m]$ is the set of these *top* elements and $M = [m]$ is the set of remaining *middle* elements. Call a pair (x, W) a *crucial pair* if $x \in M$, $W \subset M - \{x\}$, and $|W| \leqslant k$.

Proposition 2.4. *If there exists a t-collection Σ of linear orders on M such that for every crucial pair (x, W) there exist at least $k+1-|W|$ linear orders $L \in \Sigma$ such that $x > W$ in L, then* $\dim(1,k;n) \leqslant t$.

Proof. Extend each $L \in \Sigma$ to a linear order L' on $[n]$ so that $T = \{\max_{L'}[n]\colon L \in \Sigma\}$ and $T \setminus \{\max_{L'}[n]\} < M$ in L' and set $\Sigma' = \{L'\colon L \in \Sigma\}$. We claim that Σ' is a realizer of $B(1,k;n)$. Consider any k-set $Y \subset [n] \setminus \{x\}$ and let $W = Y \cap M$. Then $W < x$ in at least $k+1-|W|$ of the linear orders in Σ'. In at least one of these linear orders $Y < x$. $\square$

Theorem 2.5. *Let θ, k, n, and t be natural numbers with $\theta \geqslant 2$.*
1. *If* $\mathrm{pack}(\theta, (\theta-1)k+1, t) \geqslant n - t$, *then* $\dim(1,k;n) \leqslant t$.
2. *If* $\mathrm{pack}(\theta, \lceil [(\theta-1)k+1]/2 \rceil, \lfloor t/2 \rfloor) \geqslant n - t$, *then* $\dim(1,k;n) \leqslant t$.

Proof. First consider (1). Let $\mathscr{F} = \{F_i\colon i \in [m]\}$ be a $(\theta, (\theta-1)k+1, t)$-packing. Let $\Sigma = \{L_j\colon j \in [t]\}$ be a collection of linear orders on M such that $x < y$ in L_j if $j \in F_y \setminus F_x$. Then for any crucial pair (x, W),

$$\left| F_x \setminus \bigcup_{y \in W} F_y \right| \geqslant (\theta-1)k + 1 - (\theta-1)|W| \geqslant k + 1 - |W|.$$

Thus there exist at least $k+1-|W|$ linear orders $L \in \Sigma$ such that $x > W$ in L. So we are done by Proposition 2.4.

For (2), let $\mathscr{F} = \{F_i\colon i \in [m]\}$ be a $(\theta, \lceil [(\theta-1)k+1]/2 \rceil, \lfloor t/2 \rfloor)$-packing. Let $\Sigma = \{L_j^0, L_j^1\colon j \in [\lfloor t/2 \rfloor]\}$ be a collection of linear orders on M such that $x < y$ in L_j^b if $j \in F_y \setminus F_x$ and $b \in \{0,1\}$. Furthermore, let $x < y$ in L_j^0 iff $y < x$ in L_j^1 for all $j \in F_x \cap F_y$. Consider a crucial pair (x, W). For each $y \in W$, there exist at most $\theta - 1$ pairs (j, b) such that $y \in F_x$ and $x < y$ in L_j^b. Thus there are at least $2\lceil [(\theta-1)k+1]/2 \rceil - (\theta-1)|W| \geqslant k+1-|W|$ linear orders $L \in \Sigma$ such that $x > W$ in L and we are done by Proposition 2.4. $\square$

Corollary 2.6. *Let k, m, and n be a natural numbers such that $2 \leqslant m \leqslant \sqrt{n}$ and $k < \lfloor n/m \rfloor + m - 2$. Then* $\dim(1,k;n) \leqslant n - m$.

Proof. Set $s = \lfloor n/m \rfloor + m - 2$. Since $k < s$, $\lfloor s/2 \rfloor \geqslant \lceil (k+1)/2 \rceil$ and so by Theorem 2.5, it suffices to show that $\mathrm{pack}(2, \lfloor s/2 \rfloor, \lfloor (n-m)/2 \rfloor) \geqslant m$. Let $\Delta = \lfloor s/2 \rfloor - m + 1$. Since $m \leqslant \sqrt{n}$, $\lfloor n/m \rfloor \geqslant m$ and so $\Delta \geqslant 0$. By Lemma 1.2, $\mathrm{pack}(2, m-1+\Delta, \binom{m}{2} + m\Delta) \geqslant m$. Thus it suffice to check that $\binom{m}{2} + m\Delta \leqslant \lfloor (n-m)/2 \rfloor$.

$$\begin{aligned}\binom{m}{2} + m\Delta &= \frac{m^2-m}{2} + m\left(\left\lfloor \frac{\lfloor n/m \rfloor + m - 2}{2} \right\rfloor - m + 1\right)\\ &= m\left\lfloor \frac{\lfloor n/m \rfloor - 1}{2} \right\rfloor\\ &\leqslant \left\lfloor \frac{n-m}{2} \right\rfloor. \qquad \square\end{aligned}$$

2.3. Dushnik's theorem

We can now calculate the exact value of $\dim(1,k;n)$ for any $k \geqslant 2\sqrt{n} - 2$.

Theorem 2.7 (Dushnik). *Let k and n be positive integers such that $2\sqrt{n} - 2 \leqslant k \leqslant n - 1$. Then* $\dim(1,k;n) = n - m$, *where m is the smallest integer such that $0 \leqslant m$ and $\lfloor n/(m+1) \rfloor + m - 1 \leqslant k$.*

Proof. By Corollary 2.2, $n - m \leqslant \dim(1,k;n)$. By the minimality of m, $k < \lfloor n/m \rfloor + m - 2$. Since $2\sqrt{n} - 2 \leqslant k$, $m < \sqrt{n}$, and so by Corollary 2.6, $\dim(1,k;n) \leqslant n - m$.

Below $2\sqrt{n} - 2$ the gap between our lower and upper bounds quickly becomes significant. The following example shows that it is already $O(n)$ when $k = (2/\sqrt{3})\sqrt{n}$. $\square$

Example 2.1. *Let p be a prime. Then* $p^2 < \dim(1, 2p-1; 3p^2) \leqslant 2p^2$.

Proof. The lower bound follows from Theorem 2.1 by setting $s = p$. The upper bound follows immediately from Theorem 2.5 and Lemma 1.2

$$\mathrm{pack}\left(2, \frac{2p-1+1}{2}, \frac{2p^2}{2}\right) \geqslant p^2 \geqslant 3p^2 - p^2. \qquad \square$$

3. Very small k

A good amount of effort has been put into calculating $\dim(1,2;n)$. Spencer [14] proved that

$$\lg\lg n \leqslant \dim(1,2;n) \leqslant \lg\lg n + (\tfrac{1}{2} + o(1)) \lg\lg\lg n.$$

Trotter strengthened the lower bound to match the upper bound. Very recently Hoşten and Morris [7] have given the following characterization of $\dim(1,2;n)$ that yields a very nice proof of this result. Call an antichain $\mathscr{A}$ in $B(t)$ *good* if $S\cup T\neq[t]$, for all $S,T\in\mathscr{A}$. Let $a(n)$ be the least integer t such that $B(t)$ has at least n good antichains. Hoşten and Morris proved the following theorem, which will follow immediately from Lemmas 3.4 and 3.6.

Theorem 3.1 (Hoşten and Morris). *For every positive integer n,* $\dim(1,2;n)=a(n)+1$.

Corollary 3.2. *For every positive integer n,*

$$\dim(1,2;n)=\lg\lg n+(\tfrac{1}{2}+o(1))\lg\lg\lg n.$$

Proof. The power set of $\binom{[t]}{\lfloor (t-1)/2\rfloor}$ is a collection of good antichains in $B(t)$. By Lemma 1.3, $B(t)$ has at most $2^{(1+O((\log t)/t))\binom{t}{\lfloor t/2\rfloor}}$ antichains. Thus if $t=a(n)$ then

$$2^{\binom{[t]}{\lfloor (t-1)/2\rfloor}}\leqslant n\leqslant 2^{(1+O((\log t)/t))\binom{t}{\lfloor t/2\rfloor}}.$$

So $t=\lg\lg n+(\frac{1}{2}+o(1))\lg\lg\lg n$. □

3.1. Lower bounds

Fix a realizer $\Sigma=\{L_i\colon i\in[d]\}$ of $B(1,k;n)$ and let $<_i$ denote the order relation of L_i. For distinct $x,y\in[n]$, let $S(x,y)=\{i\in[d-1]\colon x<_i y\}$ and $\mathscr{A}_x=\mathrm{MAX}(\{S(x,y)\colon x<_d y\})$.

Proposition 3.3. *For all subsets $X=\{x_0<_d x_1<_d\cdots<_d x_k\}\in\binom{[n]}{k+1}$ and all s with $0\leqslant s\leqslant k-1$*

$$(P(s))\qquad \bigcap_{r<s} S(x_r,x_s)\setminus\bigcup_{r>s} S(x_s,x_r)\neq\emptyset.$$

(*Here the empty intersection is $[d]$ and the empty union is the empty set.*)

Proof. Since Σ is a realizer, there exists $i\in[d]$ such that $X\setminus\{x_s\}<_i x_s$. Since $s<k$, $i\neq d$. So $i\in\bigcap_{r<s}S(x_r,x_s)\setminus\bigcup_{r>s}S(x_s,x_r)$. □

Lemma 3.4. *For all integers $n>1$, $a(n)+1\leqslant\dim(1,2;n)$.*

Proof. Let $\Sigma=\{L_i\colon i\in[d]\}$ be as above. We claim that $\mathscr{A}=\{\mathscr{A}_x\colon x\in[n]\}$ is a collection of antichains witnessing that $a(n)\leqslant d-1$. Clearly each $\mathscr{A}_x$ is an antichain. For all $x_0<_d x_1<_d x_2$, $S(x_0,x_1)\cup S(x_0,x_2)\neq[d-1]$, since otherwise $P(0)$ does not hold. So $\mathscr{A}_x$ is good. Next, we show that $\mathscr{A}_{x_0}\neq\mathscr{A}_{x_1}$, for $x_0,x_1\in[n]$ with $x_0<_d x_1$. There exists $S\in\mathscr{A}_{x_0}$ such that $S(x_0,x_1)\subset S$. If $\mathscr{A}_{x_0}=\mathscr{A}_{x_1}$, then $S\in\mathscr{A}_{x_1}$, and so there exists $x_2\in[n]$ with $x_1<_d x_2$ such that $S=S(x_1,x_2)$. But then $S(x_0,x_1)\subset S(x_1,x_2)$, which violates $P(1)$. □

3.2. Upper bounds

We are going to generalize the notion of lexicographical order. Let $L=(U,\prec)$ be any linearly ordered set and define the *lexicographical order* Lex_L on 2^U by $A<B$ in Lex_L iff $\min_L(A \triangle B) \in B$. So if we identify the subsets $A \subset U$ with binary vectors v_A such that the ith coordinate of v_A is 1 iff $u_i \in A$, then $A<B$ in Lex_L iff $v_A(i)<v_B(i)$, where i is the L-least coordinate such that $v_A(i) \neq v_B(i)$. It is easily checked that Lex_L is a linear extension of $B(n)$ for any linear order L of $[n]$. For any subset $S \subset U$, let $Lex_L(S)$ be the *lexicographical order with respect to* S defined by $A<B$ in $Lex_L(S)$ iff $\min(A \triangle B) \in (B \cap \bar{S}) \cup (A \cap S)$. So $Lex_L(S)$ is the lexicographical order obtained by reversing the natural order on $\{0,1\}$ for all the coordinates in S. Then $Lex_L = Lex_L(\emptyset)$ and $Lex_L(U) = Lex_L^*$, the inverse of Lex_L. When L is clear from the context, we may just write $Lex(S)$ for $Lex_L(S)$. We will often code elements $x \in [n]$ by subsets $F_x \subset U$ using a one-to-one correspondence $x \mapsto F_x$. In this case $Lex(S)$ *induces* a linear order on $[n]$ by $x<y$ iff $F_x<F_y$ in $Lex(S)$.

Theorem 3.5 (Spencer). *For all positive integers k and n,* $\dim(1,k;n) \leqslant k2^k \log \lg n$.

Proof. Let $t=\lceil \lg n \rceil$, $U=[t]$ and $x \mapsto F_x$ be a one-to-one correspondence between $[n]$ and $2^{[t]}$. We will construct a family $\mathscr{S}$ of subsets of $[t]$ such that $|\mathscr{S}| \leqslant k2^k \log \lg n$ and $\Sigma = \{Lex(S)\colon S \in \mathscr{S}\}$ induces a realizer of $B(1,k;n)$. For this to succeed, for each pair (W,x) with $W \in \binom{[n]}{k}$ and $x \in [n] \backslash W$, we need a set $S \in \mathscr{S}$ such that for all $y \in W$,

$$(P(W,x)) \qquad \min(F_x \triangle F_y) \in S \Leftrightarrow \min(F_x \triangle F_y) \in F_y.$$

Then $W<x$ in the order induced by $Lex(S)$. Let $C=\{\min(F_x \triangle F_y)\colon y \in W\}$. In order to meet the condition $P(W,x)$ it suffices to insure that there exists $S \in \mathscr{S}$ such that $S \cap C = \{\min(F_x \triangle F_y)\colon \min(F_x \triangle F_y) \in F_y \text{ and } y \in W\}$. So to meet all the conditions $P(W,x)$ it suffices to insure that for all $C \in \binom{[t]}{k}$ and all $D \in 2^C$ there exists $S \in \mathscr{S}$ such that

$$(Q(C,D)) \qquad S \cap C = D.$$

We will obtain $\mathscr{S}$ by a probabilistic construction. Let $\mathscr{S}=\{S_i\colon i \in [d]\}$ be a random family of $d=k2^k \log t$ subsets of $[t]$. So for each $i \in [d]$ and $j \in [t]$ the probability that $j \in S_i$ is $\frac{1}{2}$. There are $2^k\binom{n}{k}$ pairs (C,D) and the probability that $Q(C,D)$ fails for a particular pair (C,D) is at most $(1-2^{-k})^d$. Thus the probability that we fail for some pair is at most

$$2^k \binom{t}{k} (1-2^{-k})^d < t^k e^{-2^{-k}d} < 1.$$

So the probability of success is greater than 0. □

Hoşten and Morris refined this argument for the case $k=2$ by putting more structure on the set U.

Lemma 3.6. *For all integers* $n>1$, $\dim(1,2;n)\leqslant 1+a(n)$.

Proof. Choose a collection $\mathscr{A}=\{\mathscr{A}_x\colon x\in[n]\}$ of good antichains in $B(d-1)$ witnessing that $a(n)\leqslant d-1$. We will construct a d-set Σ of linear orders on $\mathscr{A}$ that induces a realizer of $B(1,2;n)$. Let $L(U,\prec)$ be a linear order on $U=2^{[d-1]}$ such that $Q\subset R\Rightarrow R\prec Q$. Each antichain $\mathscr{A}_x$ is a subset of U. For $i\in[d-1]$, let $X_i=\{X\in U\colon i\in X\}$. Finally set $\Sigma=\{Lex_L(X_i)\colon i\in[d-1]\}\cup\{Lex_L(U)\}$. We shall denote the order relation induced by $Lex_L(U)$ by $<_d$ and the order relation induced by $Lex_L(X_i)$ by $<_i$. Thus $x<_i y$ iff either $i\in Q$ and $Q\in\mathscr{A}_x$ or $i\notin Q$ and $Q\in\mathscr{A}_y$, where $Q=\min_L(\mathscr{A}_x\triangle\mathscr{A}_y)$.

Consider $W'=\{x<_d y<_d z\}\in[n]$. For each $w\in W'$ we must show that there exists $i\in[d]$ such that $W'\backslash\{w\}<_i w$. If $w=z$ then $\{x,y\}<_d z$. Next suppose that $w=y$. Let $Q=\min(\mathscr{A}_x\triangle\mathscr{A}_y)$ and $R=\min(\mathscr{A}_z\triangle\mathscr{A}_y)$. Then $Q\in\mathscr{A}_x\backslash\mathscr{A}_y$ and $R\in\mathscr{A}_y\backslash\mathscr{A}_z$, since $x<_d y$ and $y<_d z$. It suffices to show that $Q\backslash R\neq\emptyset$, since $\{x,z\}<_i y$, for any $i\in Q\backslash R$. First suppose that $Q\prec R$. Then $Q\backslash R\neq\emptyset$, by the choice of L. Next suppose that $R\prec Q$. Then $R\in\mathscr{A}_x$ since Q is the first coordinate where $\mathscr{A}_x$ and $\mathscr{A}_y$ differ. Since $\mathscr{A}_x$ is an antichain, $Q\backslash R\neq\emptyset$. So the claim is true and there exists $i\in Q\backslash R$. Finally, suppose that $w=x$. Let $Q=\min(\mathscr{A}_x\triangle\mathscr{A}_y)$ and $R=\min(\mathscr{A}_x\triangle\mathscr{A}_z)$. Then $Q,R\in\mathscr{A}_x$. Since $\mathscr{A}_x$ is good, $[d-1]\backslash(Q\cup R)\neq\emptyset$, and so $\{y,z\}<_i x$, for any $i\in[d-1]\backslash(Q\cup R)$. This completes the proof. □

3.3. *A general lower bound*

So far the best lower bound we have for $\dim(1,k;n)$ when k is small is the trivial bound $\dim(1,2;k)\leqslant\dim(1,k;n)$. In this subsection we introduce a more general technique.

Theorem 3.7 (Kierstead). *Let k, m, n, and t be positive integers with* $m=n/2\binom{t}{t2^{-k+2}}$. *If* $t\leqslant\lg n$ *and* $2^{k-2}\leqslant t<2^{k-2}\dim(1,2;m)$, *then* $t<\dim(1,k;n)$.

Proof. Let $\Sigma=\{L_i\colon i\in[t]\}$ be a collection of linear orders on $[n]$ and let $<_i$ be the order relation of L_i. We must show that Σ is not a realizer of $B(1,k;n)$. For all $x\in[n]$ we will construct sets $\emptyset=W_x^0\subset W_x^1\subset\cdots\subset W_x^{k-2}\subset[n]$ and $\Sigma=\Sigma_x^0\supset\Sigma_x^1\supset\cdots\supset\Sigma_x^{k-2}$ recursively such that

1. $|W_x^s|=s$; and
2. $W_x^s\not<_i x$, for all $i\notin\Sigma_x$;

Suppose that we have constructed W_x^s and Σ_x^s. Choose w such that the set $S(w)=\{i\in\Sigma_x^s\colon w<_i x\}$ is as small as possible. Set $W_x^{s+1}=W_x^s\cup\{w\}$ and $\Sigma_x^{s+1}=S(w)$. It suffices to show that there exists a subset $X\subset[n]$ of size m and a set $S\subset\Sigma$ with $|S|\leqslant t2^{-k+2}$ such that $\Sigma_x^{k-2}\subset S$, for all $x\in X$: Since $t2^{-k+2}<\dim(1,2;m)$, there exists $x,y,z\in X$ such that $\{y,z\}\not<_i x$, for all $i\in S$. By the construction, $W_x^{k-2}\not<_i x$, for all $i\in\overline{\Sigma_x^{k-2}}$. Thus $W_x^{k-2}\cup\{y,z\}\not<_i x$, for all $i\in\Sigma$, since $\Sigma=S\cup\overline{\Sigma_x^{k-2}}$. So Σ is not a realizer of $B(1,k;n)$.

We still must prove the existence of X and S. At stage s let $G^s=\{x\in[n]\colon |\Sigma_x^s|\leqslant t2^{-s}\}$ be the set of *good* elements. We claim that $|G^s\backslash G^{s+1}|\leqslant\binom{t}{t2^{-s}}$. For each $x\in G^s$ choose a set $S_x\in\binom{[t]}{t2^{-s}}$ such that $S_x^s\supset\Sigma_x^s$. Define a digraph $D=(G^s,A)$ on G^s by $(x,y)\in A$ iff $|\{i\in S_x^s\colon y<_i x\}|\leqslant t2^{-s-1}$. Define an equivalence relation $\equiv_s$ on G^s by $x\equiv_s y$ iff $S_x^s=S_y^s$. Note that if $x\equiv_s y$, then either $(x,y)\in A$ or $(y,x)\in A$. Thus each equivalence class has at most one *bad* element. Since there are at most $\binom{t}{t2^{-s}}$ equivalence classes we have proved the claim. Thus

$$|G^{k-2}|\geqslant n-\sum_{s=1}^{k-2}\binom{t}{t2^{-s}}\geqslant n-\frac{2^t}{\sqrt{t}}\lg t\geqslant\frac{n}{2}.$$

By the pigeon hole principal one of the equivalence classes of $\equiv_{k-2}$ has size $m=n/(2\binom{t}{t2^{-k+2}})$. Let X be this class and $S=S_x^{k-2}$, where x is any element of X. $\square$

Corollary 3.8. *Let k and n be positive integers with n sufficiently large. If $k<\lg\lg n-\lg\lg\lg n$, then $2^{k-2}\lg\lg n<\dim(1,k;n)$.*

4. Intermediate values of k

We now have an exact formula for $\dim(1,k;n)$ when k is at least $2\sqrt{n}-2$ and tight bounds for $\dim(1,k;n)$ when $k<\lg\lg n-\lg\lg\lg n$. At both these boundaries there is a major change in the growth rate of $\dim(1,k;n)$ which we now explore.

4.1. Lower bounds

Next, we combine the proof techniques of Theorems 2.1 and 3.7 along with a new idea to obtain useful lower bounds on $\dim(1,k;n)$ for all k.

Theorem 4.1 (Kierstead). *Let h, k, n, k, and θ be positive integers with*

$$k=2h+\lg\lg n\quad\text{and}\quad\theta=\frac{\lg n}{2(2\lg h+\lg\lg n)}.$$

Then $\frac{1}{2}\theta h^2<\dim(1,k;n)$.

Proof. Let $\Sigma=\{L_i\colon i\in[t]\}$ be a collection of linear orders on $[n]$ and let $<_i$ be the order relation of L_i. We must show that Σ is not a realizer of $B(1,k;n)$ when $t=\theta h^2$. We construct sets W_x^s and Σ_x^s as in the proof of Theorem 3.7 except that we start with $s=h$. For each $x\in[n]$ choose W_x^h such that $|W_x^h|=h$ and $S(W_x^h)=\{i\in[t]\colon W_x^h<_i x\}$ is as small as possible. Set $\Sigma_x^h=S(W_x^h)$. Let $G^h=\{x\in[n]\colon |\Sigma_x^h|\leqslant t/(h+1)\}$. Then, $|G^h|\geqslant[n]-h$, since as in the proof of Theorem 2.1, for any $S\in\binom{[n]}{h+1}$ there exists $x\in S$ such that $|\{i\in[t]\colon S\backslash\{x\}<_i x\}|\leqslant t/(h+1)$. For $i\in[h]$, let $G^{h+i}=\{x\in[n]\colon |\Sigma_x^{h+i}|\leqslant(t/h)-i(\theta/2)\}$. We claim that $|G^{h+i}\backslash G^{h+i+1}|\leqslant t^\theta$. Suppose that $x,y\in G^{h+i}$. If

$|\Sigma_x^{h+i} \cap \Sigma_y^{h+i}| \geqslant \theta$, then either

$$|\{i \in \Sigma_x^{h+i}\colon W_x^{h+i} \cup \{y\} <_i x\}| \leqslant |\Sigma_x^{h+i}| - \frac{\theta}{2}$$

or

$$|\{i \in \Sigma_y^{h+i}\colon W_y^{h+i} \cup \{x\} <_i y\}| \leqslant |\Sigma_y^{h+i}| - \frac{\theta}{2}.$$

Thus either x or y is in G^{h+i+1}. So $\{W_x^{h+i}\colon x \in G^{h+i} \backslash G^{h+i+1}\}$ is a θ-packing. By Lemma 1.2 $|G^{h+i} \backslash G^{h+i+1}| \leqslant t^{\theta}$, as claimed. Thus $|G^{2h-2}| \geqslant n - h - (h-2)t^{\theta} \geqslant n/2$ and $|\Sigma_x^{2h-2}| \leqslant \theta$ for every $x \in G^{2h-2}$. For each $x \in G^{2h-2}$ choose $S_x \in \binom{t}{\theta}$ such that $S_x \supset \Sigma_x^{2h-2}$. Then there exists a set $X \subset G^{2h-2}$ such that $|X| \geqslant n/2\binom{t}{\theta} \geqslant \sqrt{n}$. Since $\dim(1, \lg\lg n, \sqrt{n}) \geqslant \theta$, Σ is not a realizer of $B(1,k;n)$.

Corollary 4.2. *If* $k > \lg\lg n$, *then*

$$\frac{k^2 \lg n}{16 \lg k} < \dim(1,k;n).$$

4.2. Upper bounds

Next, we consider two upper bounds. The first is better when $k < 2^{\sqrt{\lg n}}$; the second is better for $k > 2^{\sqrt{\lg n}}$.

Theorem 4.3 (Furedi and Kahn). *For all positive integers* $k < n$, $\dim(1,k;n) \leqslant (k+1)^2 \log n$.

Proof. Let $\Sigma = \{L_i\colon i \in [t]\}$ be a random collection of linear orders on $[n]$ with order relation $<_i$. For any pair (W,x) with $W \in \binom{[n]}{k}$ and $x \in [n] \backslash W$, the probability that $W \not<_i x$ is $k/(k+1)$. Thus the probability that $W \not<_i x$ for all $i \in [t]$ is $(k/(k+1))^t$. There are $\binom{n}{k}(n-k)$ choices for the pair (W,x). Thus, the probability that Σ is not a realizer of $B(1,k;n)$ is at most

$$\binom{n}{k}(n-k)(k/(k+1))^t < n^{k+1} \mathrm{e}^{-t/(k+1)} < 1,$$

provided that $t = (k+1)^2 \log n$. $\square$

Theorem 4.4 (Kierstead). *For all positive integers* $k < n$,

$$\dim(1,k;n) \leqslant 2k^2 \frac{\lg^2 n}{\lg^2 k}.$$

Proof. Set $\theta = (\lg n)/(\lg k)$. By Theorem 2.5 it suffices to check that $\operatorname{pack}(\theta, k\theta, 2k^2\theta^2) \geqslant n$. This follows easily from Lemma 1.2 since there exists a prime p such that $k\theta < p < 2k\theta$. $\square$

4.3. Summary

In the range that we do not know the exact value of $\dim(1,k;n)$, we have the following estimates.

Theorem 4.5. *Let k and n be positive integers with $k<2\sqrt{n}-2$. Then*

$$c_1 2^k \lg\lg n < \dim(1,k;n) < c_2 k 2^k \lg\lg n \quad \text{if } 2 \leqslant k < \lg\lg n - \lg\lg\lg n,$$

$$c_1 \frac{k^2 \lg n}{\lg k} < \dim(1,k;n) < c_2 k^2 \lg n \quad \text{if } \lg\lg n < k \leqslant 2^{\sqrt{\lg n}},$$

$$c_1 \frac{k^2 \lg n}{\lg k} < \dim(1,k;n) < c_2 k^2 \frac{\lg^2 n}{\lg^2 k} \quad \text{if } 2^{\sqrt{\lg n}} < k < 2\sqrt{n}-2.$$

5. Infinite cardinals

For an infinite cardinal κ, let $B(<\omega;\kappa)$ be the ordered set on the finite subsets of κ ordered by inclusion and let $\dim(<\omega;\kappa)$ be its dimension. More generally, for any cardinal α, let $B(<\alpha;\kappa)$ be the ordered set on the subsets of κ of size less than α ordered by inclusion and $\dim(<\alpha;\kappa)$ be the dimension of $B(<\alpha;\kappa)$. The infinitary version of Proposition 1.1 also holds.

Proposition 5.1. *The dimension $\dim(<\alpha;\kappa)$ of $B(<\alpha;\kappa)$ is the least cardinal τ such that there exists a collection $\{L_i\colon i\in[\tau]\}$ of τ linear orders on $[\kappa]$ with order relation $<_i$ such that $(*)$ for all $S\subset[n]$ and $x\in[n]\backslash S$ there exists $i\in[\tau]$ with $S<_i x$.*

Let $\lg\kappa$ be the least cardinal λ such that $\kappa\leqslant 2^{\lambda}$. In this section we show that $\dim(<\omega;\kappa)=\lg\lg\kappa$.

5.1. Lower bound

We use the arrow notation $\kappa\to(3)^2_\tau$ to mean that for any function $f:\binom{\kappa}{2}\to\tau$, there exists a 3-subset $X\subset\kappa$ such that f is constant on $\binom{X}{2}$. We shall need the following very weak form of the Erdős–Rado theorem (e.g. see [3, Corollary 7.5]).

Lemma 5.2. *For every infinite cardinal τ, $(2^\tau)^+\to(3)^2_\tau$.*

We are now ready to prove our lower bound that even holds for $B(<3;\kappa)$.

Lemma 5.3 (Kierstead and Milner). *For any infinite cardinal κ,*

$$\lg\lg\kappa\leqslant\dim(<3;\kappa).$$

Proof. Suppose for a contradiction that $\Sigma=\{L_\alpha: \alpha\in\tau\}$ be a realizer of $B(<3;\kappa)$, where $\tau<\lg\lg\kappa$. Let $\leqslant_\alpha$ be the order relation of L_α. Define $f:\binom{\kappa}{2}\to 2^\tau$ by $f(\{x<y\}) =g$, where $g:\tau\to 2$ is defined by $g(\alpha)=1$ if $x<_\alpha y$ and $g(\alpha)=0$ if $y<_\alpha x$. Since $\tau<\lg\lg\kappa$, $(2^{2^\tau})^+\leqslant\kappa$. Thus by Lemma 5.2 there exists a set $X=\{x<y<z\}\subset\kappa$ such that g is constant on $\binom{X}{2}$. If $g(x,y)=1$, then $\{x,y\}<_\alpha z$ for all $\alpha<\tau$. If $g(x,y)=0$, then $\{y,z\}<_\alpha x$ for all $\alpha<\tau$. In either case, $\{x,z\}\not<y$ for any $\alpha<\tau$, and so Σ is not a realizer of $B(1,2;\kappa)$. $\square$

5.2. Upper bound

A family of subsets $\mathscr{F}$ is *independent* if for all finite subfamilies $\mathscr{A}\subset\mathscr{F}$ and $\mathscr{B}\subset\mathscr{F}$ with $\mathscr{A}\cap\mathscr{B}=\emptyset$, $\cap\mathscr{A}\setminus\cup\mathscr{B}\neq\emptyset$. We shall need the following well-known lemma (e.g. see [12, Appendix]).

Lemma 5.4. *For every infinite cardinal λ there exists an independent family $\mathscr{F}\subset 2^\lambda$ with $|\mathscr{F}|=2^\lambda$.*

Lemma 5.5 (Kierstead and Milner). *For every infinite cardinal κ, $\dim(<\omega;\kappa)\leqslant \lg\lg\kappa$.*

Proof. Let $\lambda=\lg\kappa$ and $\mu=\lg\lambda$. Let $\mathscr{F}$ be an independent family of subsets of μ with $|\mathscr{F}|=\lambda$. Fix one-to-one correspondences $\alpha\mapsto F_\alpha$ between λ and $\mathscr{F}$ and $x\mapsto\mathscr{A}_x$ between κ and $2^{\mathscr{F}}$. Let $\prec$ be any well ordering on $\mathscr{F}$. Equivalently, the elements of κ are coded by binary sequences whose coordinates are elements of $\mathscr{F}$. For $\beta\in\mu$, let $\mathscr{S}_\beta=\{F\in\mathscr{F}: \beta\in F\}$. We claim that $\Sigma=\{Lex(\mathscr{S}_\beta): b\in\mu\}$ induces a realizer of $B(1,\omega;\kappa)$. Consider a finite set $S\subset\kappa$ and an element $x\in\kappa\setminus S$. For each $y\in S$, let $F_y=\min_\prec(\mathscr{A}_x\triangle\mathscr{A}_y)$. Let $\mathscr{A}=\{F_y: y\in S \text{ and } F_y\in\mathscr{A}_y\}$ and $\mathscr{B}=\{F_y: y\in S \text{ and } F_y\in \mathscr{A}_x\}$. Since $\mathscr{F}$ is independent, there exists $\gamma\in\cap\mathscr{A}\setminus\cup\mathscr{B}$. Then $S<x$ in the order induced by $Lex(\mathscr{S}_\gamma)$. $\square$

5.3. Larger α

The author and Milner also considered the problem of $\dim(<\alpha;\kappa)$ for $\omega<\alpha<\kappa^+$. They proved the following theorem.

Theorem 5.6 (Kierstead and Milner). *If $\alpha\leqslant\mu=\lg\lg(\kappa)$ are infinite cardinals, then $\mu\leqslant d(<\alpha,\kappa)\leqslant\mu^{<\alpha}$.*

Corollary 5.7. *Assume GCH. Let α and κ be cardinals with $2<\alpha\leqslant\kappa^+$.*
1. *If κ is singular then $\dim(<\alpha;\kappa)=\kappa$.*
2. *If $\lambda=\lg(\kappa)$ and $\mu=\lg\lg(\kappa)$ is regular, then*

$$\dim(<\alpha;\kappa)=\begin{cases}\mu & \text{if } 2<\alpha\leqslant\mu,\\ \lambda & \text{if } \alpha=\lambda,\\ \kappa & \text{if } \alpha\in\{\kappa,\kappa^+\}.\end{cases}$$

The simplest open question under the assumption *GCH* is whether

$$\dim(<\aleph_1;\aleph_{\omega+1})=\aleph_\omega \text{ or } \aleph_{\omega+1}?$$

6. Larger *j*

In this section we consider $\dim(j,j+s;n)$ for s small compared to n. No nontrivial lower bounds are known, so we will concentrate on upper bounds. One of the main complications in dealing with this case is that Proposition 1.1 is no longer available. Thus we must work with linear extensions of $B(j,j+s;n)$. An ordered pair of subsets $(S,T)\in\binom{[n]}{j}\times\binom{[n]}{j+s}$ is called a *critical* pair if $S\not\subset T$.

Theorem 6.1 (Brightwell et al. [1]). *For positive integers j, s, and n with $j+s<n$, $\dim(j,j+s;n)\leqslant\dim(1,2s;n)+18\log n$.*

Proof. We shall construct a realizer $\Sigma=\Sigma_s\cup\Sigma_l$, with $|\Sigma_s|=\dim(1,2s;n)$ and $|\Sigma_l|=18\log n$ such that for any critical pair (S,T), if $|S\triangle T|\leqslant 3s$ then there exists $L\in\Sigma_s$ such that $T<S$ in L and if $|S\triangle T|>3s$ then there exists $L\in\Sigma_l$ such that $T<S$ in L. Let Σ' be a collection of linear orders of size $\dim(1,2s;n)$ that forms a realizer of $B(1,2s;n)$. Let $\Sigma_s=\{Lex_{L^*}: L\in\Sigma'\}$. Suppose that (S,T) is a critical pair with $|S\triangle T|\leqslant 3s$. Then there exists $x\in S\backslash T$ and $|T\backslash S|\leqslant 2s$. Since Σ' is a realizer of $B(1,2s;n)$, there exists $L\in\Sigma'$ such that $T\backslash S<x$ in L. Thus $x<T\backslash S$ in L^* and so $T<S$ in $Lex_{L^*}\in\Sigma_s$.

It remains to construct Σ_l. Let $x\mapsto\sigma_x$ be a one-to-one correspondence between $[n]$ and $[3s]^{[t]}$, where $t=3\log n$. We claim that we can choose $x\mapsto\sigma_x$ so that for every $S\in\binom{[n]}{3s}$ there exists $j\in[t]$ such that $|\{\sigma_x(j): x\in S\}|>s$: For each $x\in[n]$ and $j\in[t]$ choose $\sigma_x(j)\in[3s]$ randomly. Then

$$\Pr(|\{\sigma_x(j): x\in S\}|\leqslant s)<\binom{3s}{s}3^{-3s}<3^{-s},\quad \forall j\in[t],\ \forall S\in\binom{[n]}{3s},$$

$$\Pr(\forall j\in[t], |\{\{\sigma_x(j): x\in S\}\}|\leqslant s)<3^{-st},\quad \forall S\in\binom{[n]}{3s},$$

$$\Pr\left(\exists S\in\binom{[n]}{3s}\forall j\in[t], |\{\sigma_x(j): x\in S\}|\leqslant s\right)<\binom{n}{3s}3^{-st}<1.$$

Thus the probability of failure is less than one. For a $j\in[t]$, $a\in[3s]$, and $S\subset[n]$, set $S_{j,a}=\{x\in S: \sigma_x(j)=a\}$. Choose linear extensions M_1 and M_2 of $B(n)$ such that $|S|<|T|$ implies that $S<T$ in both M_1 and M_2 and furthermore $|S|=|T|$ implies that $S\leqslant T$ in M_1 iff $T\leqslant S$ in M_2. For $i\in[2]$, $j\in[t]$, and $a\in[3s]$, let $L_{i,j,a}$ be any linear extension of the partial order $P_{i,j,a}$ on $B(n)$ defined by $S<T$ in $P_{i,j,a}$ iff $S\subsetneq T$ or $S_{j,a}<T_{j,a}$ in M_i. It is easily checked that $P_{i,j,a}$ is indeed a partial order. Set $\Sigma_l=\{L_{i,j,a}: i\in[2], j\in[t], a\in[3s]\}$. Then $|\Sigma_l|=18\log n$. Now suppose that (S,T) is a critical pair with $|S\triangle T|>3s$. Then there exists $j\in[t]$ such that $|D_j|>s$, where $D_j=\{\sigma_x(j): x\in$

$S \triangle T\}$. It follows that there exists $a \in D_j$ such that $|T_{j,a}| \leqslant |S_{j,a}|$. Then there exists $i \in [2]$ such that $T < S$ in $L_{i,j,a}$. □

Corollary 6.2. *For positive integers j, s, and n with $j + s < n$,*

$$\dim(j, j+s; n) \leqslant O(s^2 \lg n).$$

Kostochka has improved this bound in the case that $s = 1$.

Theorem 6.3 (Kostochka). *For positive integers j and n with $j + 1 < n$,*

$$\dim(j, j+1; n) = O\left(\frac{\lg n}{\lg \lg n}\right).$$

Proof. Let $x \mapsto \sigma_x$ be a one-to-one correspondence between $[n]$ and Perm $([t])$, where $t = 2(\lg n)/(\lg \lg n)$. Let M_i for $i \in [2]$ be defined as above. For each $S \subset [n]$, set $S_{j,a} = \{x \in S\colon \sigma_x(j) = a\}$. For each $a \in [t]$ and $i \in [2]$, define a partial order $P_{i,a}$ on $B(n)$ as follows. Consider $S, T \subset [n]$. Let j be least element of $[t]$ such that $S_{j,a} \triangle T_{j,a} \neq \emptyset$, if it exists; otherwise j is undefined. Then $S < T$ in $P_{i,a}$ iff $S \subsetneq T$ or $S_{j,a} < T_{j,a}$ in M_i. It is easily checked that $P_{i,a}$ is a partial order. Let $L_{i,a}$ be any linear extension of $P_{i,a}$ and set $\Sigma = \{L_{i,a}\colon (i,a) \in [2] \times [t]\}$. Then $|\Sigma| = 4(\lg n)/(\lg \lg n)$. We claim that Σ is a realizer of $B(j, j+s; n)$. Consider a critical pair (S, T). Let j be the least element of $[t]$ such that $|D_j| > 1$, where $D_j = \{\sigma_j(x)\colon x \in S \triangle T\}$. Such a j exists since $x \mapsto \sigma_x$ is a one-to-one correspondence. It follows that there exists $a \in D_j$ such that $|T_{j,a}| \leqslant |S_{j,a}|$. Then there exists $i \in [2]$ such that $T < S$ in $L_{i,a}$. □

6.1. Very large s

Several authors have considered the problem of calculating $\dim(j, n-j; n)$. For $j < \frac{1}{8} n^{1/3}$, i.e., j small and $s = n - 2j$ large, the following theorems give exact answers.

Theorem 6.4 (Hurlbert et al. [6]). *For all $n \geqslant 5$,*

$$\dim(2, n-2; n) = n - 1.$$

Theorem 6.5 (Furedi). *For all positive integers j and n such that $3 \leqslant j$ and $250 j^3 < n$, $\dim(j; n-j) = n - 2$.*

It is easily seen that if $j \leqslant j' \leqslant k' \leqslant k < n' \leqslant n$ then $\dim(j', k'; n') \leqslant \dim(j, k; n)$. It follows that

$$\dim(1, k-1; n-1) \leqslant \dim(2, k; n) \leqslant \dim(1, k; n).$$

It follows from Dushnik's results that for large k, this gap is at most 1 and often 0. Hurlbert, Kostochka, and Telysheva strengthened this to prove the following Dushnik-type theorem.

Theorem 6.6 (Hurlbert et al. [6]). *Let n and m be positive integers with $5 \leqslant n$, $m \leqslant \sqrt{n}$, and $3 \leqslant \lfloor (n-1)/m \rfloor$.*

Let

$$\left\lfloor \frac{n}{m+1} \right\rfloor + m - 1 \leqslant k - 1 < \left\lfloor \frac{n-1}{m} \right\rfloor + m - 2.$$

1. *If* $n - 1 \not\equiv 0 \bmod m$, *then* $\dim(1, k-1; n-1) = n - m - 1 = \dim(2, k; n)$.
2. *If* $n - 1 \equiv 0 \bmod m$, *then* $\dim(1, k-1; n-1) = n - m - 1 \leqslant \dim(2, k; n) \leqslant n - m$.

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